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Deformed algebras from the inverse Schwinger method

Kyung-Hyun Cho† and S U Park ‡

† Department of Physics, Chonbuk National University, Chonju, 506–756, Korea

‡ Department of Physics, Jeonju University, Chonju, 560–759, Korea

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Abstract. We consider a problem which may be viewed as inverse to the Schwinger realization of Lie algebra, and suggest a procedure for deforming the algebra so obtained. We illustrate the method through a few simple examples extending Schwinger's $su(1, 1)$ construction. As results, various q -deformed algebras are (re-)produced as well as their undeformed counterparts. Some extensions of the method are briefly pointed out.

1. Introduction

The Schwinger method of realizing $su(2)$ algebra in terms of the products of two independent sets of Heisenberg–Weyl (or oscillator, boson) algebras [1] is a typical example of the oscillator realization of Lie algebras. This method has been extended to the classical (super) Lie algebras by employing various types of oscillator algebras such as Fermi, para-Bose, and para-Fermi algebras as well as Heisenberg–Weyl algebras, and has been a valuable tool for the representation theory of Lie algebras [2].

Recently, quantum deformation of a Lie algebra [3, 4] based on the deformation of a Lie–Poisson structure has been actively studied due to its prominent roles in diverse areas of physics and mathematics (for a review see [5], for example). The deformation has also been studied in various different approaches, such as in the form of a pseudogroup [6], as the transformation group of a non-commutative geometry [7, 8], and by using filtration [9]. As in the classical Lie algebra theory, many works have been done to realize the (super)quantum groups *à la* Schwinger method, by introducing various q -deformed oscillator algebras starting with the $su_q(2)$ realization in terms of q -oscillators in [10–12]. In particular, much attention has been paid to the generalized deformed Bose or Fermi oscillators of single-mode ([13–15] and references therein) or multi-modes [16] in itself in connection with some physical applications as well as with the quantum group.

In this paper we propose considering a problem which is in a sense inverse to the Schwinger method as explained below, and a natural procedure of deforming the algebra so obtained. As shown in the text, various q -deformed algebras are produced together with new ones, and, moreover, many interesting extensions will be possible, as commented in the last section.

Suppose an algebra is given whose generators are factorized into two independent pairs of operators, with one part taken as a Heisenberg–Weyl algebra for simplicity. The problem we address is to determine the other pair of operators that satisfy the given algebra, which will be called an inverse problem to the Schwinger realization. In section 2 we will state the problem more precisely together with our working assumptions, following closely the

Schwinger construction of $su(1, 1)$ algebra as a typical example. The solution is obtained by solving the defining commutation relations algebraically considering the full Hilbert space, which will be referred to as an *algebraic* solution.

Next, as a natural means to deform the algebraic solution, the following procedure is suggested. We take expectation values on the Hilbert space of the Heisenberg–Weyl algebra in the defining commutation relations of the given algebra, and find the operator solution obeying the resulting *effective* commutation relations. This solution will be called a *deformed* solution and form a deformed algebra of the one obtained in the inverse Schwinger problem, since it has one more parameter (due to the averaging) than the algebraic solution which acts like a deformation parameter, and it reduces to the algebraic one at a special value of the parameter.

In section 3 we illustrate the above ideas with several specific examples. With a few simplifying assumptions in the first three subsections, a variety of algebras such as exponential phase operator (q -oscillator), Heisenberg–Weyl algebra (Calogero–Vasiliev algebra), and $su(1, 1)$ algebra ($su(1, 1)$ -Calogero–Vasiliev algebra) emerge as algebraic (deformed) solutions, respectively. In the last subsection we give an example in which the algebraic solution is absent, while the deformed one is an interesting q -deformed algebra which becomes a q -oscillator or a $su_q(1, 1)$ algebra depending on the parameters.

In the final section, we give some comments and discuss the extensions of the method. Some simple formulae related with the expectation values are given in the appendix for convenience.

2. Inverse Schwinger method

We start by recalling the Schwinger construction of $su(1, 1)$ algebra as an example, in terms of two sets of independent oscillators $\{(a, a^\dagger, n_a), (b, b^\dagger, n_b)\}$ which satisfy the Heisenberg–Weyl algebras as in (A1). For a given $su(1, 1)$ algebra,

$$[J_-, J_+] = 2J_0 \quad [J_0, J_\pm] = \pm J_\pm \quad (2.1)$$

the Schwinger method realizes the algebra in the product forms of oscillators as follows:

$$\begin{aligned} J_+ &= a^\dagger b^\dagger & J_- &= ba \\ J_0 &= \frac{1}{2}(n_a + n_b + 1). \end{aligned} \quad (2.2)$$

We note the following property of the Schwinger construction:

$$[n_i, J_\pm] = \pm J_\pm \quad [n_i, J_0] = 0 \quad i = a, b. \quad (2.3)$$

It is natural to think that the property (2.3) will determine the Schwinger construction of J_\pm as in (2.2), once J_0 is chosen as in (2.2). This will be an inverse procedure of the Schwinger method, so we call it the *inverse Schwinger method*.

Let us formulate the inverse Schwinger method by taking a close analogy with the above $su(1, 1)$ example. A similar work can be undertaken following the $su(2)$ construction of Schwinger as well. We introduce $\{D_-, D_+, D_0\}$ instead of $\{J_-, J_+, J_0\}$, and assume the following commutation relations resembling (2.1) and (2.3):

$$[D_-, D_+] = D_0 \quad (2.4)$$

$$[n_i, D_\pm] = \pm D_\pm \quad i = a, b \quad (2.5)$$

$$[n_i, D_0] = 0. \quad (2.6)$$

It will be natural to take D_0 as a function of n_a and n_b in view of the last commutator (2.6),

$$D_0 = D_0(n_a, n_b). \tag{2.7}$$

We note that the commutation relations between D_0 and D_{\pm} need not be specified, since they are obtained by (2.5). Then the algebra satisfies the Jacobi identity for the generators $\{n_a, n_b, D_+, D_-\}$.

We briefly consider the representation of the D -operator, which will be only formal and not explicit, since the D_0 is not specified yet. In the Hilbert space $\{|m_a, m_b\rangle\}$, where $n_i|m_a, m_b\rangle = m_i|m_a, m_b\rangle, m_i = 0, 1, 2, \dots, i = a, b$, the commutator (2.4) is reduced to the following difference equation:

$$F(m_a + 1, m_b + 1) - F(m_a, m_b) = D_0(m_a, m_b) \tag{2.8}$$

where $F(m_a, m_b)$ is the matrix element of D_+D_- . By noting that $n_a - n_b$ commutes with all the generators $\{n_a, n_b, D_+, D_-\}$, (2.8) can also be considered as a difference equation for a function of l and s , where $m_a = l + s$ and $m_b = l - s$. The inverse problem is equivalent to solving (2.8) for a given D_0 with appropriate initial conditions on $F(m_a, m_b)$.

From now on we consider D_{\pm} operators in the product form of a - and b -contribution. In order not to deviate too much from the Schwinger construction, we choose one part of D_{\pm} as an Heisenberg–Weyl generator (a, a^\dagger) , and the other part, denoted B_{\pm} (which are to be determined by (2.4) or (2.8) for a given D_0) is assumed to commute with (a, a^\dagger) . That is, we put

$$D_+ = a^\dagger B_+ \quad D_- = a B_- \tag{2.9}$$

and B_{\pm} satisfy

$$[a, B_{\pm}] = 0 \quad [a^\dagger, B_{\pm}] = 0 \tag{2.10}$$

$$[n_b, B_{\pm}] = \pm B_{\pm} \tag{2.11}$$

where the last commutator follows from (2.5).

Now, from (2.4) and (2.9) we get

$$aa^\dagger B_- B_+ - a^\dagger a B_+ B_- = D_0(n_a, n_b). \tag{2.12}$$

Therefore, within our assumptions, D_0 should be at most the first order in n_a to yield consistent algebraic solutions for B -operators,

$$D_0(n_a, n_b) = n_a G_1(n_b) + G_2(n_b). \tag{2.13}$$

Due to (2.11) we may put

$$\begin{aligned} B_+|m_b\rangle &= \sqrt{C(m_b + 1)}|m_b + 1\rangle \\ B_-|m_b + 1\rangle &= \sqrt{C(m_b + 1)}|m_b\rangle \end{aligned} \quad m_b = 0, 1, 2, \dots \tag{2.14}$$

In addition we choose $|0_b\rangle$ as the vacuum for B -operators

$$B_-|0_b\rangle = 0. \tag{2.15}$$

Then (2.12)–(2.15) lead to a difference equation,

$$\begin{aligned} C(m_b + 1) - C(m_b) &= G_1(m_b) \\ C(m_b + 1) &= G_2(m_b) \quad m_b = 1, 2, \dots \\ C(1) &= G_1(0) = G_2(0) \end{aligned} \tag{2.16}$$

This is the equation to solve for the inverse Schwinger method, and its solution will be called an *algebraic* solution. Note that this equation appears also in the investigation of

deformed single-mode oscillators in general (see, for example, [15]), although our motivation is different.

As mentioned in the introduction, we proceed to obtain the deformed algebra of the above one given by (2.16). We simply take expectation values on the Hilbert space of the a -oscillators in (2.12) using (A6) and (A7),

$$q^2(\tilde{B}_-\tilde{B}_+ - G_2(n_b)) - (\tilde{B}_+\tilde{B}_- + G_1(n_b) - G_2(n_b)) = 0 \quad (2.17)$$

where $q^2 = e^{\beta\epsilon_a}$, ϵ_a is the quantum level of the a -oscillator, and $\tilde{}$ is used to denote that the expectation values are taken for the a -oscillators. Referring to (2.11), there still holds

$$[n_b, \tilde{B}_\pm] = \pm\tilde{B}_\pm. \quad (2.18)$$

Thus putting, as in (2.14),

$$\begin{aligned} \tilde{B}_+|m_b\rangle &= \sqrt{\tilde{C}(m_b+1)}|m_b+1\rangle \\ \tilde{B}_-|m_b+1\rangle &= \sqrt{\tilde{C}(m_b+1)}|m_b\rangle \end{aligned} \quad m_b = 0, 1, 2, \dots \quad (2.19)$$

and with the choice of vacuum as in (2.15),

$$\tilde{B}_-|0_b\rangle = 0 \quad (2.20)$$

we obtain from (2.17) the following equation for the deformed solution:

$$\begin{aligned} q^2(\tilde{C}(m_b+1) - G_2(m_b)) - (\tilde{C}(m_b) + G_1(m_b) - G_2(m_b)) &= 0 \\ q^2(\tilde{C}(1) - G_2(0)) &= G_1(0) - G_2(0) \end{aligned} \quad m_b = 1, 2, \dots \quad (2.21)$$

Now we make some comments. Taking expectation values can be viewed as a thermal average physically if we imagine a system whose Hamiltonian is $h_a = \epsilon_a n_a$ and the B -operators distinguish the degenerate energy eigenstates, as in a Landau problem, for instance. Alternatively, any relevant weighted average may be taken as being associated with the system under consideration described by commuting operators a_\pm and B_\pm . Compared with (2.16), which fully takes account of the operator equation (2.12), the above (2.21) may contain only limited, or in a sense averaged, information connected to the a -oscillator. Thus, we call (2.17) or (2.21) an *effective* commutation relation of (2.12). Since the solution to (2.21) will contain the parameter q and reduce to the algebraic solution to (2.12) or (2.16) as $q \rightarrow \infty$, it will be referred to as a *deformed* solution with a deformation parameter q . In the next section we examine simple cases of the inverse Schwinger method and present the corresponding algebraic and deformed solutions as illustrations of using it to obtain deformed algebras.

3. Algebraic solutions and deformations

We restrict our attention to the following forms of $G_1(m_b)$, $G_2(m_b)$,

$$\begin{aligned} G_1(m_b) &= g(m_b+1) - g(m_b) \\ G_2(m_b) &= g(m_b+1) \end{aligned} \quad m_b = 0, 1, 2, \dots \quad (3.1)$$

in which case the difference equations (2.16), (2.21) become *homogeneous*. An example of the inhomogeneous difference equation will be given in subsection 3.4.

To obtain the algebraic solution we should consider $g(0) = 0$ in solving (2.16) in order to satisfy the initial condition in (2.16). Then the algebraic solution is given by

$$B_-B_+ = g(n_b+1) \quad B_+B_- = g(n_b). \quad (3.2)$$

The deformed solution is obtained from (2.21) as

$$\tilde{B}_- \tilde{B}_+ = g(n_b + 1) - g_0 q^{-2(n_b+1)} \quad \tilde{B}_+ \tilde{B}_- = g(n_b) - g_0 q^{-2n_b} \quad (3.3)$$

where $g_0 = g(0)$ is not restricted to zero in this case.

To avoid possible misunderstandings we stress that the solution to (2.16), which we are referring to as the algebraic solution (3.2), is the true solution to the problem we termed an inverse Schwinger problem, and exists only if $g(0) = 0$ for the choice of vacuum as in (2.15). However, the effective commutation relation (2.17), which is obtained by taking expectation values on the a -oscillators in (2.16), admits a more general solution B_{\pm} given in (3.3) sharing the same properties as B_{\pm} (compare equations (2.18), (2.19), (2.20) with (2.11), (2.14), (2.15)). In addition \tilde{B}_{\pm} are reduced to B_{\pm} as $q \rightarrow \infty$, which suggests interpreting them as deformed operators of B_{\pm} with a deformation parameter q .

In the following subsections 3.1–3.3 we examine some examples of solutions in detail with $g(m_b)$ of the form,

$$g(m_b) = c_0 + c_1 m_b + c_2 m_b^2 \quad m_b = 1, 2, \dots \quad (3.4)$$

taking $g(0)$ to be zero for the algebraic solution or as a non-zero parameter for the deformed solution, respectively, as mentioned above. This must be the simplest generalization of the Schwinger realization of the $su(1, 1)$ case which corresponds to $c_0 = c_2 = 0, c_1 = 1$, and $g(0) = 0$.

3.1. Exponential phase operator and its q -deformation

We consider the case $c_1 = c_2 = 0$ and normalize $c_0 = 1$ in (3.4). In this case (3.2) is

$$B_- B_+ = 1 \quad B_+ B_- = 1 - |0_b\rangle\langle 0_b|. \quad (3.5)$$

The explicit forms of B_{\pm} are given by (see (2.14))

$$B_- = \sum_{m_b=0}^{\infty} |m_b\rangle\langle m_b + 1| \equiv e^{i\phi} \quad (3.6)$$

$$B_+ = \sum_{m_b=0}^{\infty} |m_b + 1\rangle\langle m_b| \equiv e^{-i\phi}.$$

These have been known as Susskind and Glogower's exponential phase operators [17, 18] which are non-unitary and non-commuting. Equations (2.11) and (2.15) are

$$[n_b, e^{\mp i\phi}] = \pm e^{\mp i\phi} \quad (3.7)$$

$$e^{i\phi} |0_b\rangle = 0. \quad (3.8)$$

We note that the b -oscillators obeying the Heisenberg–Weyl algebra are expressed in the ‘polar’ decomposition form as follows by using the above exponential phase operators:

$$b = e^{i\phi} \sqrt{n_b} \quad b^\dagger = \sqrt{n_b} e^{-i\phi}. \quad (3.9)$$

Let us consider the deformed solution (3.3) given as

$$\tilde{B}_- \tilde{B}_+ = 1 - g_0 q^{-2(n_b+1)} \quad (3.10)$$

$$\tilde{B}_+ \tilde{B}_- = (1 - g_0 q^{-2n_b})(1 - |0_b\rangle\langle 0_b|).$$

If we take $g_0 = 0$, (3.5) is recovered. Thus a non-zero g_0 gives the q -deformation. Equations (2.18) and (2.17) are

$$\begin{aligned} [n_b, \tilde{B}_\pm] &= \pm \tilde{B}_\pm \\ q^2 \tilde{B}_- \tilde{B}_+ - \tilde{B}_+ \tilde{B}_- &= q^2 - 1 + (1 - g_0) |0_b\rangle \langle 0_b| \end{aligned} \quad (3.11)$$

and we recall that the vacuum is chosen as usual in (2.20). Note that when $g_0 = 1$, (3.11) is a version of the q -deformed oscillator introduced in [10].

Introducing the q -deformed number operator $\{n_b\}$ as

$$\{n_b\} = 1 - g_0 q^{-2n_b} \quad (3.12)$$

we can write the explicit form of \tilde{B}_\pm in terms of the above exponential phase operator B_\pm

$$\tilde{B}_- = e^{i\phi} \sqrt{\{n_b\}} \quad \tilde{B}_+ = \sqrt{\{n_b\}} e^{-i\phi}. \quad (3.13)$$

This form is essentially equivalent to that of the b -oscillator (3.9) with changes only in the 'amplitude' part by replacing the normal number operator of the b -oscillator into the q -number operator $\{n_b\}$. Thus the Hilbert space for the \tilde{B} -operators is the same as that of the exponential phase operator. Since the \tilde{B}_\pm reduce to the exponential phase operator as $q \rightarrow \infty$, they may be considered as a q -deformation of the exponential phase operators.

It is clear from the above that q -deformation alters the amplitude part of the operator leaving the phase part, the exponential phase operator, unchanged, which is a well known fact. This feature shows up repeatedly in the following subsections, as in the second equations of (3.5), (3.10), (3.11). It does not cause any difficulty in writing a polar decomposition form for a deformed operator as seen in (3.13). However, we will frequently pay attention only to a case where the commutation relation takes a simple form (e.g. take $g_0 = 1$ in (3.11)).

3.2. Oscillator and Calogero-Vasiliev oscillator

Let us consider the case where $c_2 = 0$, $c_1 = 1$ in (3.4). The algebraic solution (3.2) is

$$\begin{aligned} B_- B_+ &= n_b + 1 + c_0 \\ B_+ B_- &= (n_b + c_0)(1 - |0_b\rangle \langle 0_b|). \end{aligned} \quad (3.14)$$

Using the exponential phase operator (3.6) introduced in the previous subsection, B_\pm are

$$B_- = e^{i\phi} \sqrt{c_0 + n_b} \quad B_+ = \sqrt{c_0 + n_b} e^{-i\phi}. \quad (3.15)$$

When $c_0 = 0$ these B_\pm become normal b -oscillators and correspond to the Schwinger realization of $su(1, 1)$.

The deformed solution (3.3) is given by

$$\begin{aligned} \tilde{B}_- \tilde{B}_+ &= n_b + 1 + c_0 - g_0 q^{-2(n_b+1)} \\ \tilde{B}_+ \tilde{B}_- &= (n_b + c_0 - g_0 q^{-2n_b})(1 - |0_b\rangle \langle 0_b|). \end{aligned} \quad (3.16)$$

The \tilde{B}_\pm can also be written in the polar form, as in section 3.1, with changes occurring in the amplitude part only.

Considering the case $c_0 = g_0$ (see the remarks at the end of section 3.1), the commutator for \tilde{B}_\pm is given by

$$[\tilde{B}_-, \tilde{B}_+] = 1 - c_0 (q^{-2} - 1) q^{-2n_b} \quad (3.17)$$

with $[n_b, \tilde{B}_\pm] = \pm \tilde{B}_\pm$. This algebra is a q -deformation of a usual Heisenberg-Weyl algebra which is recovered as $q \rightarrow \infty$.

Let us note that if we put formally $q^{-2} = -1$, the algebra (3.17) is the Calogero–Vasiliev† algebra: we change the notation $\tilde{B}_-, \tilde{B}_+, c_0$ into C, C^\dagger, ν , respectively. Then we have

$$CC^\dagger = n_b + 1 + \nu + \nu K \quad C^\dagger C = n_b + \nu - \nu K, \tag{3.18}$$

where the parity operator K is given by

$$\begin{aligned} K &= (-1)^{n_b} & K^2 &= 1 \\ KC &= -CK & KC^\dagger &= -C^\dagger K. \end{aligned} \tag{3.19}$$

The explicit forms of C, C^\dagger , are given by

$$C = e^{i\phi} \sqrt{n_b + \nu - \nu K} \quad C^\dagger = \sqrt{n_b + \nu - \nu K} e^{-i\phi}. \tag{3.20}$$

Thus a Calogero–Vasiliev algebra with the deforming parameter ν is obtained

$$\begin{aligned} [C, C^\dagger] &= 1 + 2\nu K \\ [n_c, C] &= -C & [n_c, C^\dagger] &= C^\dagger \end{aligned} \tag{3.21}$$

where the number operator n_c is defined as

$$n_c = \frac{1}{2}(CC^\dagger + C^\dagger C - 1) = n_b + \nu. \tag{3.22}$$

3.3. Holstein–Primakoff and Calogero–Vasiliev extension of $su(1, 1)$

Let us consider the algebraic solution (3.2) for $c_2 = 1$. In this case the solution is

$$\begin{aligned} B_- B_+ &= (n_b + 1)(n_b + 1 + c_1) + c_0 \\ B_+ B_- &= (n_b(n_b + c_1) + c_0)(1 - |0_b\rangle\langle 0_b|). \end{aligned} \tag{3.23}$$

Specializing to the case $c_0 = 0$, B_\pm together with B_0 defined as

$$B_0 = n_b + \frac{1}{2}(c_1 + 1) \tag{3.24}$$

realize the $su(1, 1)$ algebra known as the Holstein–Primakoff realization [27],

$$\begin{aligned} B_- &= b\sqrt{B_0 + \frac{1}{2}(c_1 - 1)} \\ B_+ &= \sqrt{B_0 + \frac{1}{2}(c_1 - 1)} b^\dagger \\ [B_-, B_+] &= 2B_0 & [B_0, B_\pm] &= \pm B_\pm. \end{aligned} \tag{3.25}$$

The deformed solution (3.3) is obtained as

$$\begin{aligned} \tilde{B}_- \tilde{B}_+ &= (n_b + 1)(n_b + 1 + c_1) + c_0 - g_0 q^{-2(n_b+1)} \\ \tilde{B}_+ \tilde{B}_- &= (n_b(n_b + c_1) + c_0 - g_0 q^{-2n_b})(1 - |0_b\rangle\langle 0_b|). \end{aligned} \tag{3.26}$$

Let us consider the case $c_0 = g_0$. Then \tilde{B}_\pm together with \tilde{B}_0 defined as

$$\tilde{B}_0 = n_b + \frac{1}{2}(c_1 + 1 + c_0(1 - q^{-2})q^{-2n_b}) \tag{3.27}$$

realize a q -deformed $su(1, 1)$ algebra (not in a standard form),

$$[\tilde{B}_-, \tilde{B}_+] = 2\tilde{B}_0 \quad [B_0, \tilde{B}_\pm] = \pm \tilde{B}_\pm \tag{3.28}$$

† This is the simplest case of the extended Heisenberg algebra involving exchange operators [19–21] which is widely used in the study of the integrable Calogero–Moser–Sutherland-type many-body systems. It is also equivalent to the single-level para-bose algebra [22–24] and its q -deformation is performed in [25, 26].

and reduce to the above Holstein–Primakoff realization (3.25) as $q \rightarrow \infty$.

In this case if we put formally $q^{-2} = -1$, as in the previous subsection, we find an interesting deformed $su(1, 1)$ algebra realized by the Calogero–Vasiliev oscillator C, C^\dagger introduced in (3.18)–(3.22): let $c_1 = 2c_0 + 1$. We change the notations of \tilde{B}_\pm, B_0, c_0 into J_\pm^c, J_0^c, ν , respectively. Their explicit forms are

$$\begin{aligned} J_-^c &= C\sqrt{n_c + 1 + \nu K} \\ J_+^c &= \sqrt{n_c + 1 + \nu K}C^\dagger \\ J_0^c &= n_c + 1 \end{aligned} \tag{3.29}$$

where the parity operator K and the number operator n_c of the Calogero–Vasiliev oscillator are given in equations (3.19), (3.22). The aforementioned deformed $su(1, 1)$ is obtained as

$$\begin{aligned} [J_-^c, J_+^c] &= 2J_0^c + 2\nu K & [J_0^c, J_\pm^c] &= \pm J_\pm^c \\ J_\pm^c K &= -K J_\pm^c & J_0^c K &= K J_0^c \end{aligned} \tag{3.30}$$

which reduce to the Holstein–Primakoff realization (3.25) of the $su(1, 1)$ algebra by an ordinary oscillator when $\nu = 0$.

3.4. q -Holstein–Primakoff realization of $su_q(1, 1)$

As an example of a more complicated case of the inverse Schwinger method, let us consider D_0 which leads to an inhomogeneous difference equation from (2.12). We take D_0 as

$$D_0(n_a, n_b) = \theta((n_a + \frac{1}{2} - \lambda)(\mu + \frac{1}{2} - n_a)) \tag{3.31}$$

where $\theta(x)$ is the Heaviside step function and μ, λ are integers of $0 \leq \lambda \leq \mu$. The function takes unity as its value in the finite interval $\lambda \leq n_a \leq \mu$ with $(\mu - \lambda + 1)$ -independent states.

The equation (2.12) has no solution for above D_0 ; however, on taking expectation values on the a -oscillator it admits a non-trivial solution. Using (A5), the expectation value of the step function (3.31) is given by

$$\langle \theta((n_a + \frac{1}{2} - \lambda)(\mu + \frac{1}{2} - n_a)) \rangle = q^{-2\lambda}(1 - q^{-2(\mu - \lambda + 1)}) \tag{3.32}$$

and the resulting effective commutation relation is

$$q^2 \tilde{B}_- \tilde{B}_+ - \tilde{B}_+ \tilde{B}_- = (q^2 - 1)q^{-2\lambda}(1 - q^{-2(\mu - \lambda + 1)}) \tag{3.33}$$

and $[n_b, \tilde{B}_\pm] = \pm \tilde{B}_\pm$ from (2.18).

With the vacuum as in (2.20), the solution to (3.33) is easily obtained and related with the q -deformed operators given in section 3.1 by a simple normalization. When we take

$$\mu \rightarrow \infty \quad \lambda = n_b \tag{3.34}$$

the solution to (3.33) is also related to the q -deformed operators of section 3.1 by a n_b -dependent transformation.

Another interesting case is

$$\mu = 2(n_b + \sigma) - 1 \quad \lambda = 0. \tag{3.35}$$

Transforming $\{\tilde{B}_\pm, n_b\}$ into new generators $\{\tilde{J}_\pm, J_0\}$ related by

$$\begin{aligned} \tilde{J}_- &= (q^2 - 1)^{-1} \tilde{B}_- q^{n_b + \sigma + 1/2} \\ \tilde{J}_+ &= (q^2 - 1)^{-1} q^{n_b + \sigma + 1/2} \tilde{B}_+ \\ J_0 &= n_b + \sigma \end{aligned} \tag{3.36}$$

then $\{\tilde{J}_\pm, J_0\}$ satisfy the $su_q(1, 1)$ algebra:

$$[\tilde{J}_-, \tilde{J}_+] = [2J_0] \quad [J_0, \tilde{J}_\pm] = \pm \tilde{J}_\pm \tag{3.37}$$

where $[x] = (q^x - q^{-x})/(q - q^{-1})$ is a normal q -number. Explicitly,

$$\begin{aligned} \tilde{J}_- &= e^{i\phi} \sqrt{[n_b][n_b + 2\sigma - 1]} = b_q \sqrt{[n_b + 2\sigma - 1]} \\ \tilde{J}_+ &= \sqrt{[n_b][n_b + 2\sigma - 1]} e^{-i\phi} = \sqrt{[n_b + 2\sigma - 1]} b_q^\dagger \end{aligned} \tag{3.38}$$

with the standard q -oscillators b_q, b_q^\dagger , give the q -Holstein–Primakoff realization of $su_q(1, 1)$ in terms of a q -oscillator [28, 29].

4. Discussion

We have considered an inverse problem to the well known Schwinger realization of Lie algebras. Taking Schwinger’s $su(1, 1)$ construction by two commuting oscillators as our model case, we have made a few assumptions to explain the ideas in a simple setting as given in sections 2 and 3. However, various interesting examples such as the exponential phase operators, normal oscillators, and the Holstein–Primakoff realization of $su(1, 1)$ algebra appear as solutions to the inverse problem we considered, although they have been known from other motivations.

More importantly, we have obtained the corresponding q -deformed algebras naturally by solving the effective commutation relations. Therefore, on the basis of simple but non-trivial examples examined in section 3, we may regard the procedure taken in this paper as an efficient way of obtaining deformed algebras. It will be worthwhile to understand the mathematical structure involved in this procedure. In particular, the q -Heisenberg–Weyl algebra of the subsection 3.1 has been explained on the basis of the contact metric structure of the Heisenberg–Weyl group manifold [30].

If we extend the above inverse Schwinger method to the fermionic algebra [31], as in sections 2 and 3, we can easily find the various fermionic q -deformed algebras. Extension to the multi-mode case is also possible. As a simplest case, one may find a set of independent oscillators and corresponding q -oscillators. More non-trivial q -deformed systems, such as the $SU_q(N)$ -covariant system of q -oscillators, will appear through considerations similar to those in subsection 3.4. Further detailed description is in preparation, and will be reported.

Appendix

We consider a single-mode oscillator with generators $\{n_a, a^\dagger, a\}$ obeying the following familiar commutation relations ;

$$[a, a^\dagger] = 1 \quad [n_a, a] = -a \quad [n_a, a^\dagger] = a^\dagger \tag{A1}$$

where $n_a = a^\dagger a$. The Hilbert space for the oscillator is spanned by the number eigenstates $|m_a\rangle, m_a = 0, 1, 2, \dots$. We take the Hamiltonian h_a

$$h_a = \epsilon_a n_a = \epsilon_a a^\dagger a \tag{A2}$$

where ϵ_a is its quantal energy level. The expectation value of an operator Φ is defined by

$$\langle \Phi \rangle \equiv \frac{\text{tr } e^{-\beta h_a} \Phi}{\text{tr } e^{-\beta h_a}} \tag{A3}$$

where the denominator represents the character of the oscillator algebra with $q^2 = e^{\beta\epsilon_a}$,

$$\chi = \text{tr} e^{-\beta h_a} = \frac{1}{1 - q^{-2}}. \quad (\text{A4})$$

We present some expectation values for later use:

$$\langle \theta(n_a + \frac{1}{2} - \alpha) \rangle = e^{-\alpha\beta\epsilon_a} = q^{-2\alpha} \quad (\text{A5})$$

$$\langle aa^\dagger \rangle = \frac{1}{1 - e^{-\beta\epsilon_a}} = \frac{1}{1 - q^{-2}} \quad (\text{A6})$$

$$\langle a^\dagger a \rangle = \frac{e^{-\beta\epsilon_a}}{1 - e^{-\beta\epsilon_a}} = \frac{q^{-2}}{1 - q^{-2}} \quad (\text{A7})$$

where $\theta(x)$ is a step function. In (A5), when $\alpha = 0$, the expectation value of $\theta(n_a + \frac{1}{2})$ is equivalent to that of the unit.

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